

# Non- $\mathbb{Z}_2$ symmetric braneworlds in scalar tensorial gravity

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We obtain, via the Gauss-Codazzi formalism, the expression of the effective Einstein-Brans-Dicke projected equations in a non- $\mathbb{Z}_2$  symmetric braneworld scenario which presents hybrid compactification. It is shown that the functional form of such equations resembles the one in the Einstein's case, except by the fact that they bring extra informations in the context of exotic compactifications.

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## I. INTRODUCTION

Advances in the formal structure of string theory point to the emergence, and necessity, of a scalar-tensorial theory of gravity. It seems that, at least at high energy scales, the Einstein's theory is not enough to explain the gravitational phenomena [1]. In other words, the existence of a scalar (gravitational) field acting as a mediator of the gravitational interaction together with the usual purely rank-2 tensorial field is, indeed, a natural prediction of unification models as supergravity, superstrings and M-theory [2]. This type of modified gravitation was first introduced in a different context in the 60's in order to incorporate the Mach's principle into relativity [3], but nowadays it acquired different sense in cosmology and gravity theories.

On the other hand, braneworld models have being extensively studied in the recent years [4]. Such models present an elegant way to solve the hierarchy problem. In particular, in the Randall-Sundrum model, which can be understood as an effective compactification of the Horava-Witten model [5], the hierarchy problem is solved due to a warped non-factorable topology and the set up encodes two mirror 3-branes (domain walls) embedded in a five-dimensional bulk. The extra dimension, transverse to the branes, is an  $S^1/\mathbb{Z}_2$  orbifold.

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In the context of the Brans-Dicke gravity (the simplest scalar-tensor theory we have in the literature), two braneworld models were proposed using a local and a global cosmic strings (references [6] and [7], respectively) in order to establish all the bulk-brane structure. In these models the final scenario is composed by two extra dimensions, one compactified on the brane and another transverse to the brane (the so-called hybrid compactification). From the gravitational point of view, a consistent way to extract information of codimension one brane models is using the well-known Gauss-Codazzi formalism. This has been done in the context of the General Relativity Theory, with  $\mathbb{Z}_2$  symmetry [8] and without such a symmetry [9]. The result obtained in [9] is particularly interesting to treat models which present hybrid compactification. In the ref. [10] the Einstein-Brans-Dicke (EBD) effective equation on the brane were obtained with  $\mathbb{Z}_2$  symmetry. Going forward, and motivated by the results of refs. [6, 7, 9] we turn again our attention to the application of the Gauss-Codazzi formalism to braneworld models in Brans-Dicke theory but at this time without the  $\mathbb{Z}_2$  symmetry. The main purpose of this paper is to obtain the effective EBD equation in this context. This work is organized as follows: In the next Section we briefly comment the role of the  $\mathbb{Z}_2$  symmetry in braneworld models. In the Section III we obtain the form of the effective EBD equation in a very similar way of what was done to the Einstein's case [9]. And finally in the Section IV we summarize and conclude our results.

## II. THE ROLE OF THE $\mathbb{Z}_2$ SYMMETRY

For the sake of completeness we expose in brief lines some important remarks about the role of the  $\mathbb{Z}_2$  symmetry in braneworld models. The role of such a symmetry in the braneworld scenario is multiple. In the extra dimensional framework this symmetry prevents the existence of off-diagonal fluctuations on the space-time metric, as in the original Kaluza-Klein framework. Moreover, the  $\mathbb{Z}_2$  symmetry acts as an important element in the quantum effective theory on the brane; in particular, it is possible to show that with such a symmetry one obtains the right chirality of standard model fermions [11]. Apart of this, it is well-known that, if  $n_\alpha$  is an unitary vector orthogonal to the brane, this symmetry acts as a signal reversor in the jump of the  $n_\alpha$  across the brane. Therefore, it determines univocally the extrinsic curvature in the scope of the so-called Israel-Darmois matching conditions.

All these properties are desirable in the implementation of the braneworld set up and provide facility in the execution of several calculations as well. However, it is not a necessary condition in the mathematical sense. In one hand, the “Index Theorems” seem to provide a good approach to

the chirality problem [12] and, in what concerns the extrinsic curvature, a generalization for this problem has been done in the General Relativity Theory [9]. In the context of the reference [9], there is a key piece concerning the application of the Gauss-Codazzi formalism, which is the mean value of the extrinsic curvature,  $\langle K_{\mu\nu} \rangle$ . By definition, the mean value of any tensorial quantity, say  $X$ , is given by

$$\langle X \rangle = \frac{1}{2}(X^+ + X^-), \quad (1)$$

where  $X^\pm$  are the both limits of  $X$  approaching the brane from both sides  $\pm$ . It is possible to decompose  $K_{\mu\nu}$  in a traceless and a non-null trace part. The traceless part is responsible by the shear of the  $n_\alpha$  vector along the brane. The mean value of the shear, encoded in the mean value of  $K_{\mu\nu}$ , clearly vanishes if one imposes the  $\mathbb{Z}_2$  symmetry. However, in an hybrid compactification scenario, where there is at least one extra dimension compactified on the brane, the mean value of the shear term is appreciable. For an illustrative picture one can think about a simply extra dimension on the brane with  $S^1$  topology. This is the case found in the models of refs. [6, 7].

To finalize this Section, we should stress that all the modifications generated by this symmetry are encoded in the geometry of the system. The global brane topology remains the same, independently whether the extra transversal dimension is an orbifold or not. In other words, this symmetry changes locally the brane set up (e.g., it includes extra terms in the metric) but the topological properties remain the same.

### III. GAUSS-CODAZZI FORMALISM IN THE NON- $\mathbb{Z}_2$ SYMMETRIC CASE

Our goal in this Section is to generalize the Gauss-Codazzi formalism in a braneworld scenario which presents hybrid compactification in an arbitrary dimension without  $\mathbb{Z}_2$  symmetry in the Brans-Dicke Theory. After some preliminary considerations we particularize our results for the models inspired in the refs. [6, 7].

#### A. Notation and Conventions

We treat the brane as a submanifold of  $(n - 1)$ -dimensions embedded in a  $n$ -dimensional manifold, the bulk. The one-to-one correspondence between each point of the brane with the bulk points enables an induced topology on the brane, with a respective induced metric. Calling the bulk metric as  $g_{\mu\nu}$ , the brane metric is given by  $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ . Note that, according to this

equation, the brane is a time-like hypersurface. Defining, in a similar way of the eq. (1), the jump of any tensorial quantity,  $[X]$ , by

$$[X] = X^+ - X^-, \quad (2)$$

it is easy to see that

$$[AB] = \langle A \rangle [B] + [A] \langle B \rangle, \quad (3)$$

and

$$\langle AB \rangle = \langle A \rangle \langle B \rangle + \frac{1}{4} [A] [B]. \quad (4)$$

The Gauss equation, which relates the brane and the bulk Riemann tensors, is given by (denoting by  $\bar{\phantom{x}}$  quantities on the brane)

$$\bar{R}_{\mu\nu\lambda\alpha} = R_{\kappa\beta\sigma\rho} q_\mu^\kappa q_\nu^\beta q_\lambda^\sigma q_\alpha^\rho - 2K_{\mu[\alpha} K_{\lambda]\nu}, \quad (5)$$

while the Codazzi equation is

$$2\bar{\nabla}_{[\lambda} K_{\nu]\mu} = R_{\alpha\kappa\sigma\beta} n^\alpha q_\mu^\kappa q_\nu^\sigma q_\lambda^\beta. \quad (6)$$

The Weyl tensor,  $C_{\alpha\beta}^{\mu\nu}$ , is related with the bulk geometric quantities in an arbitrary dimension by

$$R_{\alpha\beta}^{\mu\nu} = C_{\alpha\beta}^{\mu\nu} + \frac{4}{n-2} g_{[\alpha}^{[\mu} R_{\beta]}^{\nu]} - \frac{2}{(n-1)(n-2)} g_{[\alpha}^{[\mu} g_{\beta]}^{\nu]} R. \quad (7)$$

The equations (5), (6) and (7) together give

$$\bar{R}_{\mu\nu} = Y_{\mu\nu} + K K_{\mu\nu} - K_\mu^\lambda K_{\lambda\nu}, \quad (8)$$

where the

$$Y_{\mu\nu} = \frac{n-3}{n-2} R_{\alpha\beta} q_\mu^\alpha q_\nu^\beta + \frac{1}{n-2} R_{\alpha\beta} q^{\alpha\beta} q_{\mu\nu} - \frac{1}{n-1} R q_{\mu\nu} + E_{\mu\nu}, \quad (9)$$

and  $E_{\mu\nu} = C_{\alpha\gamma\beta\delta} n^\gamma n^\delta q_\mu^\alpha q_\nu^\beta$ . It is important to remark that the extrinsic curvature, as well the  $Y_{\mu\nu}$  tensor belong to the hypersurface ( $Y_{\mu\nu} n^\nu = 0 = K_{\mu\nu} n^\nu$ ). Another important characteristic of the  $Y_{\mu\nu}$  tensor is that the trace  $Y$  is related with the Einstein tensor,  $G_{\mu\nu}$ , by  $Y = -2G_{\mu\nu} n^\mu n^\nu$ . Certainly, working in the Brans-Dicke Theory, this term will bring important contributions from the dilaton field (24).

### B. Obtaining the EBD effective equation on the brane

In order to project the equations on the brane, one needs to apply the operations defined in (1) and (2) in the eq. (8) taking into account the properties showed in eqs. (3) and (4). Therefore, starting with  $[\bar{R}_{\mu\nu}]$ , one has<sup>1</sup>

$$[\bar{R}_{\mu\nu}] = 0 = [Y_{\mu\nu}] + \langle K \rangle [K_{\mu\nu}] + [K] \langle K_{\mu\nu} \rangle - \langle K_{[\mu}^{\alpha} \rangle [K_{\nu]\alpha}]. \quad (10)$$

The quantities  $[K]$  and  $[K_{\mu\nu}]$  were already derived in the ref. [10] to the case of a six-dimensional bulk with a five-dimensional brane. We shall particularize the present analysis to this case from now on. The result to the jump of the extrinsic curvature was obtained for an empty-bulk model in which the stress-tensors of the bulk -  $T_{\mu\nu}$  - and the brane -  $S_{\mu\nu}$  - take the usual form<sup>2</sup>

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} + \delta S_{\mu\nu}, \quad (11)$$

$$S_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}, \quad (12)$$

where  $\Lambda$  is the bulk cosmological constant,  $\tau_{\mu\nu}$  is the stress-tensor of the brane regular matter and  $\lambda$  is the brane tension, or the vacuum energy of the brane in the case of isotropic Poincaré invariant branes. With such decompositions, the jump of the extrinsic curvature in the Brans-Dicke (BD) gravity is given by (see [10] for details)

$$[K_{\mu\nu}] = -\frac{8\pi}{\phi} \left( \tau_{\mu\nu} + \frac{q_{\mu\nu}}{2(3+2w)} ((w-1)\lambda - (w+1)\tau) \right), \quad (13)$$

and the trace is

$$[K] = \frac{8\pi(w-1)}{2\phi(3+2w)} (\tau - 5\lambda), \quad (14)$$

where  $w$  is the BD parameter and  $\phi$  is the BD scalar field (generically called here as “dilaton”).

Substituting the equations (13) and (14) in the eq. (10) one finds

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<sup>1</sup> We should not confuse the operation defined in eq. (2) with the notation used here to indicate the commutation of indices.

<sup>2</sup> It is important to note that the form of the bulk stress-tensor, with the delta term, is problematic in a complete cosmological scenario. However, for the purpose of this paper there is no problem with such a decomposition.

$$\begin{aligned}
-\left(\frac{8\pi}{\phi}\right)^{-1}[Y_{\mu\nu}] &= \langle K_{\alpha[\mu}\tau_{\nu]}^{\alpha} - \left(\tau_{\mu\nu} + \frac{q_{\mu\nu}}{2(3+2w)}((w-1)\lambda - (w+1)\tau)\right)\langle K \rangle \\
&+ \frac{(3(1-w)\lambda - (w+3)\tau)}{2(3+2w)}\langle K_{\mu\nu} \rangle.
\end{aligned} \tag{15}$$

This equation will be particularly important in order to find the mean value of the extrinsic curvature. For now, let us first derive a first form to the EBD effective equation. So, applying the mean value operator in the equation (8) we have

$$\langle \bar{R}_{\mu\nu} \rangle = \bar{R}_{\mu\nu} = \langle Y_{\mu\nu} \rangle + \frac{1}{4} \left( [K][K_{\mu\nu}] - [K_{\mu}^{\alpha}][K_{\nu\alpha}] \right) + \langle K \rangle \langle K_{\mu\nu} \rangle - \langle K_{\mu}^{\alpha} \rangle \langle K_{\nu\alpha} \rangle. \tag{16}$$

With the Ricci tensor on the brane one can contract it with  $q^{\mu\nu}$  to find the scalar curvature. The result reads

$$\bar{R} = \langle Y \rangle + \frac{1}{4} \left( [K]^2 - [K^{\mu\nu}][K_{\mu\nu}] \right) + \langle K \rangle^2 - \langle K^{\mu\nu} \rangle \langle K_{\mu\nu} \rangle, \tag{17}$$

and finally we can construct the Einstein's tensor on the brane. In order to appreciate the role of the extrinsic curvature shear pointed in the Sec. II as well as signals of deviations arising in the scope of the BD gravity, let us decompose  $Y_{\mu\nu}$  and  $K_{\mu\nu}$  as

$$Y_{\mu\nu} = \frac{Y}{5}q_{\mu\nu} + \varpi_{\mu\nu}, \tag{18}$$

and

$$K_{\mu\nu} = \frac{K}{5}q_{\mu\nu} + \zeta_{\mu\nu}, \tag{19}$$

where  $\varpi_{\mu\nu} = \frac{3}{4} \left( R_{\alpha\beta} q_{\mu}^{\alpha} q_{\nu}^{\beta} - \frac{1}{5} R_{\alpha\beta} q^{\alpha\beta} q_{\mu\nu} \right) + E_{\mu\nu}$ . The  $\zeta_{\mu\nu}$  term is the responsible by the shear of the  $n_{\mu}$  vector. With the above decompositions the effective Einstein's tensor on the brane reads

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}q_{\mu\nu}\bar{R} = -\Lambda_5 q_{\mu\nu} + G_{N5}\tau_{\mu\nu} + \pi_{\mu\nu} + \langle \varpi_{\mu\nu} \rangle + \frac{3}{5}\langle K \rangle \langle \zeta_{\mu\nu} \rangle - \langle \zeta_{\mu}^{\alpha} \rangle \langle \zeta_{\nu\alpha} \rangle, \tag{20}$$

where  $\Lambda_5$  is the effective cosmological constant on the 4-brane given by

$$\Lambda_5 = \frac{3}{10}\langle Y \rangle + \frac{6}{25}\langle K \rangle^2 - \frac{1}{2}\langle \zeta^{\alpha\beta} \rangle \langle \zeta_{\alpha\beta} \rangle + \frac{3}{8}\left(\frac{8\pi}{\phi}\right)^2 \frac{\lambda(w-1)}{(3+2w)^2}(\tau + \lambda(w-1)), \tag{21}$$

the  $G_{N5}$  term, which mimics the Newtonian gravitational constant, is

$$G_{N5} = \frac{3}{8} \left( \frac{8\pi}{\phi} \right)^2 \frac{\lambda(w-1)}{(3+2w)}, \quad (22)$$

and the  $\pi_{\mu\nu}$  term, quadratic on the brane matter stress-tensor, is given by

$$\begin{aligned} \pi_{\mu\nu} = & \left( \frac{8\pi}{\phi} \right)^2 \frac{\tau\tau_{\mu\nu}(w+3)}{8(3+2w)} - \frac{1}{4} \left( \frac{8\pi}{\phi} \right)^2 \tau_\mu{}^\alpha \tau_{\nu\alpha} + \frac{1}{8} \left( \frac{8\pi}{\phi} \right)^2 \tau^{\alpha\beta} \tau_{\alpha\beta} q_{\mu\nu} \\ & - \left( \frac{8\pi}{\phi} \right)^2 \frac{\tau^2 q_{\mu\nu}}{8(3+2w)^2} (w^2 + 3w + 3). \end{aligned} \quad (23)$$

It is time to make a pause to analyze some properties of eq. (20). The first term contains the effective cosmological constant (21). As expected for this type of models it may not be constant. We stress that the  $\langle Y \rangle$  term encodes derivative terms of the scalar field, in fact

$$\begin{aligned} \langle Y \rangle = & -2 \left( \left\langle \frac{8\pi}{\phi} T_{\mu\nu} n^\mu n^\nu \right\rangle + \left\langle \frac{w}{\phi^2} \left( \phi_{,\mu} \phi_{,\nu} n^\mu n^\nu - \frac{1}{2} \phi_{,\alpha} \phi_{,\alpha} \right) \right\rangle \right. \\ & \left. + \left\langle \frac{1}{\phi} \left( \phi_{,\mu;\nu} n^\mu n^\nu - \frac{8\pi}{3+2w} T \right) \right\rangle \right), \end{aligned} \quad (24)$$

where  $\phi_{,\mu} = \nabla_\mu \phi$ . Note that all the angled bracket terms of the equation (20) are zero under  $\mathbb{Z}_2$  symmetry, except  $\langle Y \rangle$ . Indeed, when  $\mathbb{Z}_2$  is imposed, we recover all the results found in [10]. The second term in (20) leads to the effective gravitational constant, which strongly depends on the brane tension. The factor  $\langle \varpi_{\mu\nu} \rangle$  is a generalization of the Weyl tensor in the  $\mathbb{Z}_2$  symmetric case. The terms containing  $\langle \zeta_{\nu\alpha} \rangle$  arise only due the non-symmetric scenario. In the light of equation (19), it is the mean of the shear of  $n_\mu$ . It is expected that in a models of hybrid compactification - with some compact extra dimension on the brane - the term  $\langle \zeta_{\nu\alpha} \rangle$  is appreciable, since the shear will be certainly important. Therefore, one see that the imposition of the  $\mathbb{Z}_2$  symmetry in hybrid compactification models means that some important information is lost in the projection of the EBD equation. One more remark is important here. As in the symmetric case, analyzed in the ref. [10], here there is also a type of coupling between the brane tension and the  $(w-1)$  factor. This two quantities always appear together. It can led to some cumbersome inconsistency between pure BD gravity (where the BD parameter is expected to be  $\sim 1$ ) and braneworld models. We stress that it is not a problem in the context of this work, since we are dealing with BD theory as an intermediate sector to develop and formalize braneworld models according to the tendencies pointed by string theory. Anyway, it is just an apparent incompatibility. An analysis of consistence

conditions, similar to what was done in ref. [13], to the BD case show that there is not a forbidden value to the BD parameter [14].

It is easy to note from the above equations that all that appears because of the extrinsic curvature mean value terms. Therefore, in order to completely solve the system it is necessary to express  $\langle K_{\mu\nu} \rangle$  in terms of useful quantities. We follow the work done in ref. [9] and, since we are repeating their job<sup>3</sup>, we just give the main necessary steps to isolate  $\langle K_{\mu\nu} \rangle$ .

The first part is to find an appropriate redefinition of the brane stress-tensor in terms of which the isolation of  $\langle K_{\mu\nu} \rangle$  can be done in a simplest way<sup>4</sup>. We start with a new brane stress-tensor defined by

$$\hat{\tau}_{\mu\nu} \equiv \tau_{\mu\nu} + (A\lambda + B\tau)q_{\mu\nu}, \quad (25)$$

where  $A$  and  $B$  are constants to be determined. Now, after expressing  $[K_{\mu\nu}]$  and  $[K]$  in terms of  $\hat{\tau}$  one can, for  $w \neq 1$ , find the constants<sup>5</sup>  $A = \frac{3(1-w)}{4(3+2w)}$  and  $B = -\frac{(w+3)}{4(3+2w)}$  in such way that the eq. (10) gives

$$0 = [Y_{\mu\nu}] + \langle K \rangle [K_{\mu\nu}] + \frac{8\pi}{\phi} \langle K_{[\mu}{}^{\alpha} \rangle \hat{\tau}_{\nu]\alpha}. \quad (26)$$

The quantities are now expressed in terms of  $\hat{\tau}_{\mu\nu} = \tau_{\mu\nu} + \frac{(3(1-w)\lambda - (w+3)\tau)}{4(3+2w)}q_{\mu\nu}$  by

$$[K_{\mu\nu}] = -\frac{8\pi}{\phi} \left( \hat{\tau}_{\mu\nu} - \frac{\hat{\tau}}{3} q_{\mu\nu} \right), \quad (27)$$

and the trace is

$$[K] = \frac{8\pi}{\phi} \frac{2\hat{\tau}}{5}. \quad (28)$$

Treating  $\hat{\tau}_{\mu\nu}$  as a  $5 \times 5$  symmetric matrix with  $\det(\hat{\tau}_{\mu\nu}) \neq 0$  we have, by definition,  $(\hat{\tau}^{-1})^{\mu\sigma} \hat{\tau}_{\mu\nu} = \delta_{\nu}^{\sigma}$ . Then, inserting the eq. (27) into (26) and multiplying the result by  $(\hat{\tau}^{-1})^{\mu\nu}$  one has

$$\frac{8\pi}{\phi} \langle K \rangle = \frac{3(\hat{\tau}^{-1})^{\mu\nu} [Y_{\mu\nu}]}{9 - (\hat{\tau}^{-1})_{\mu}^{\mu} \hat{\tau}_{\nu}^{\nu}}. \quad (29)$$

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<sup>3</sup> We refer the reader to the ref. [9] for the details of the calculations in the Einstein's case. Here, we are just generalizing for the Brans-Dicke case.

<sup>4</sup> Note that from the equation (15) it is not easy to isolate  $\langle K_{\mu\nu} \rangle$ .

<sup>5</sup> In the case analyzed in ref. [9] those constants are (for a  $n$ -dimensional bulk)  $A = -\frac{(n-3)}{2(n-2)}$  and  $B = \frac{1}{2(n-2)}$ .



To guarantee a finite mean for the trace of the extrinsic curvature one must impose<sup>6</sup> that  $9 - (\hat{\tau}^{-1})^\mu_\mu \hat{\tau}^\nu_\nu \neq 0$ . In fact, there exist many examples of ordinary matter which stress-tensor is related with the dimension of the manifold in question, e. g., topological defects and Chern-Simons term (for odd dimensions). However, up to our knowledge there is not an approach which circumvents this problem. Here we assume that  $9 \neq (\hat{\tau}^{-1})^\mu_\mu \hat{\tau}^\nu_\nu$ . Substituting the eq. (29) into (26) we arrive at

$$-\frac{8\pi}{\phi} \langle K_{[\mu}{}^\alpha \rangle \hat{\tau}_{\nu]\alpha} = [Y_{\mu\nu}] + \frac{3(\hat{\tau}^{-1})^{\alpha\beta} [Y_{\alpha\beta}]}{9 - (\hat{\tau}^{-1})^\sigma_\sigma \hat{\tau}^\gamma_\gamma} (-\hat{\tau}_{\mu\nu} + \frac{\hat{\tau}}{3} q_{\mu\nu}), \quad (30)$$

or, in a more compact way,

$$\frac{8\pi}{\phi} \langle K_{[\mu}{}^\alpha \rangle \hat{\tau}_{\nu]\alpha} \equiv -[\hat{Y}_{\mu\nu}]. \quad (31)$$

In order to completely isolate the  $\langle K_{\mu\nu} \rangle$  term solving the equation above, one has to use the vielbein decomposition. To do so, it is convenient to work with a complete basis, say  $h_\mu^{(i)}$  ( $i = 0, 1, \dots, 4$ ), of orthonormal vectors constructed by the contraction of an orthonormal matrix set which represents a local Lorentz transformation and turns  $\hat{\tau}_{\mu\nu}$  (and consequently  $\tau_{\mu\nu}$ ) diagonal. The orthonormality conditions are given by

$$\begin{aligned} h^\mu_{(i)} h_{\mu(j)} &= \eta_{(i)(j)}, \\ \sum_{i,j=0}^4 \eta_{(i)(j)} h_\mu^{(i)} h_\nu^{(j)} &= \sum_{j=0}^4 h_\mu^{(j)} h_{\nu(j)} = q_{\mu\nu}, \end{aligned} \quad (32)$$

where  $\eta_{(i)(j)}$  is the Minkowski metric and we do not assume Einstein's summation convention over the tangent indices ( $i$ ). In terms of this frame, the diagonal  $\hat{\tau}_{\mu\nu}$  reads

$$\hat{\tau}_{\mu\nu} = \sum_i \hat{\tau}_{(i)} h_\mu^{(i)} h_{\nu(i)}. \quad (33)$$

Plugging the eq. (33) in (31) and contracting the result with  $h^\mu_{(i)} h^\nu_{(j)}$  we arrive, after some algebra, at

$$\frac{8\pi}{\phi} \langle K_{\mu\nu} \rangle = - \sum_{i,j} \frac{h_\mu^{(i)} h_\nu^{(j)}}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} [\hat{Y}_{(i)(j)}], \quad (34)$$

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<sup>6</sup> In general, for a  $n$ -dimensional bulk this constraint reads  $(n-3)^2 \neq (\hat{\tau}^{-1})^\mu_\mu \hat{\tau}^\nu_\nu$ .

where  $[\hat{Y}_{(i)(j)}] \equiv h^\mu_{(i)} h^\nu_{(j)} [\hat{Y}_{\mu\nu}]$ . The equation above is the main goal of the vielbein decomposition application. It allows us to write the mean of the extrinsic curvature in a suitable isolated way. It is easy to note that the diagonal term of (34) is given by

$$\sum_{i=j} \frac{h_\mu^{(i)} h_\nu^{(j)}}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} [\hat{Y}_{(i)(j)}] = \frac{1}{2} (\hat{\tau}^{-1})_\mu{}^\alpha [\hat{Y}_{\alpha\nu}], \quad (35)$$

then we can write down the matching condition to the mean value of extrinsic curvature

$$\begin{aligned} \frac{8\pi}{\phi} \langle K_{\mu\nu} \rangle &= \frac{1}{2} (\hat{\tau}^{-1})_\mu{}^\alpha [Y_{\alpha\nu}] + \frac{3(\hat{\tau}^{-1})^{\beta\gamma} [Y_{\beta\gamma}]}{2(9 - (\hat{\tau}^{-1})^\sigma{}_\rho \hat{\tau}^\rho{}_\sigma)} \left( q_{\mu\nu} - \frac{\hat{\tau}^\rho{}_\rho (\hat{\tau}^{-1})_{\mu\nu}}{3} \right) \\ &- \sum_{i \neq j} \frac{h_\mu^{(i)} h_\nu^{(j)}}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} [\varpi_{(i)(j)}], \end{aligned} \quad (36)$$

where, following the standard notation,  $[\varpi_{(i)(j)}] \equiv h^\mu_{(i)} h^\nu_{(j)} [\varpi_{\mu\nu}]$ . From the equation (36) one can find the expressions for  $\langle K \rangle$  and  $\langle \zeta_{\mu\nu} \rangle$  from (19).

In order to complete the analysis, let us write down the effective EBD projected equation on the brane from eq. (20). In the orthonormal frame the diagonal term reads

$$\bar{G}_{(i)(i)} = -\Lambda_5 + G_{N5} \tau_{(i)} + \pi_{(i)} + \langle \varpi_{(i)(i)} \rangle + \frac{3}{5} \langle K \rangle \langle \zeta_{(i)(i)} \rangle + \sum_k \langle \zeta_{(i)}^{(k)} \rangle \langle \zeta_{(i)(k)} \rangle, \quad (37)$$

where  $\pi_{(i)} = \frac{1}{4} \left( \frac{8\pi}{\phi} \right)^2 \left( \frac{(w+3)}{2(3+2w)} \tau \tau_{(i)} - \tau_{(i)}^2 + \frac{1}{2} \left( \sum_j \tau_{(j)}^2 \right) - \frac{(w^2+3w+3)}{2(3+2w)^2} \tau^2 \right)$ . Moreover, there exist off-diagonal terms in the Einstein's tensor given by

$$\bar{G}_{(i)(j)} = \langle \varpi_{(i)(j)} \rangle + \frac{3}{5} \langle K \rangle \langle \zeta_{(i)(j)} \rangle + \sum_k \langle \zeta_{(i)}^{(k)} \rangle \langle \zeta_{(j)(k)} \rangle. \quad (38)$$

The new off-diagonal terms in the eq. (38) arise in the non- $\mathbb{Z}_2$  symmetric context only. These last two equations summarize the generalization of the Gauss-Codazzi formalism to braneworld models without  $\mathbb{Z}_2$  symmetry in the Brans-Dicke gravity. We shall comment these results in the next Section.

#### IV. CONCLUDING REMARKS

In this work our main goal was to generalize the Gauss-Codazzi formalism to non- $\mathbb{Z}_2$  symmetric braneworlds in the framework of Brans-Dicke gravity. It is a direct generalization of our previous

work (see, please, ref. [10]). The analysis with such tools are very general and can be applied in a wide range of models. We must stress, however, that some attention should be paid to the topological scenario of the model. The Gauss-Codazzi formalism is quite useful if we are dealing with codimension one braneworld models, which is in fact the case analyzed here. Moreover, in order to keep a compact internal space without orbifolding it, the extra dimension can be endowed with a  $S^1$  topology (which is the more natural choice) or a bounded  $\mathbb{R}$  line. In the context of braneworld models, if we choose the bounded  $\mathbb{R}$  line, the cut-offs on the extra dimension are understood as new branes [6, 7, 15]. From a phenomenological point of view, the multiple brane scenario is potentially interesting, see for instance ref. [16]. However, from the mathematical point of view, the most natural choice would be the compact  $S^1$  space.

The off-diagonal terms of the EBD effective equation on the brane can extract more information in the case of anisotropic types of braneworlds. If one studies braneworlds via Gauss-Codazzi formalism within the scope of  $\mathbb{Z}_2$  symmetry, the off-diagonal terms are suppressed of the brane metric. However, in a hybrid compactification model such off-diagonal fluctuations naturally arises (the Kaluza-Klein original idea). Besides, in the cosmological context, off-diagonal terms in the energy-momentum tensor usually describe anisotropic matter sources. As one can see from equations (37) and (38), under the imposition of  $\mathbb{Z}_2$  symmetry, the angled brackets become zero and we lose any information about the anisotropic terms.

The presence of the dilaton field in the effective EBD equation on the brane can lead to important phenomenological implications. In particular, models which constrain the dependence of such field to the transverse dimension are very interesting, since the projected equations give the possibility of subtle but important deviations from the General Relativity Theory in the study of many gravitational systems [17]. In this vein, the cosmological analysis of such models seems to be a very promising field of research. More specifically, the study of the scalar field influence in cosmological systems, targeting a systematic comparison with braneworld models in General Relativity and with  $\mathbb{Z}_2$  symmetry, is quite important in order to give parameters for future developments in braneworlds. We shall to refer to those questions in the future. We stress, however, that the complete influence of the dilaton field is model dependent.

Finally, we do not touch here the problem of stabilizing the dilaton field because it is out of the scope of this paper. In principle, it can be obtained by the addition of a well-behaved potential in the Brans-Dicke part of the gravitational action [18]. To finalize, we stress that by the shape of the dilaton field it is possible to make a fine tuning in the value of  $\Lambda_5$  via eqs. (21) and (24). Of course, it can also be useful as an *ad hoc* argument to stabilize the distance between the branes in

a multi-brane scenario.

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